ASSIGNMENT-1

THEORY OF VIBRATIONS

SUBMITTED BY:

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TOTAL SHEETS - 1+8
Problem No. 1: A harmonic motion has amplitude of 0.2 cm and a period of 0.15 sec.

Determine the following:

a) maximum velocity
b) maximum acceleration

Solution:

The harmonic motion can be demonstrated by a mass suspended from a weightless spring as shown in the Fig. 1.1. When the mass(m) is displaced from its rest (equilibrium) position and released, it will oscillate up and down. Harmonic motion is often represented as the projection on a straight line of a point that is moving on a circle at a constant angular speed \( \omega \), the displacement \( z \) can be written as

\[
z = A \sin \omega t
\]

Where, \( A \) is the amplitude of oscillation, usually measured mm from the equilibrium position of the mass and \( \tau \) is the period of harmonic motion, usually measured in second.

Because the motion repeats itself in \( 2\pi \) radians, we have the relationship \( \omega = \frac{2\pi}{\tau} \).

Period of harmonic motion

\( \tau = 0.15 \) sec.

Amplitude of oscillation

\( A = 0.2 \) cm

Circular frequency of the motion

\[
\omega = \frac{2\pi}{\tau} = 41.89 \text{ rad/sec.}
\]

Velocity of the motion, (Using dot-notation for the derivative)

\[
v = z \cdot \dot{z} = \omega A \cos \omega t
\]

\[
v_{\text{max}} = z \cdot \dot{v}_{\text{max}} = \omega A \cos(0) = 83.78 \text{ mm/sec}
\]

Acceleration of the motion,

\[
a = z \cdot \ddot{z} = \omega^2 A \sin \omega t
\]

\[
a_{\text{max}} = z \cdot \ddot{a}_{\text{max}} = \omega^2 A \sin(\pi/2) = 3509 \text{ mm/sec}^2
\]
Problem No. 2: A mass-spring-dashpot system (W=356N) is subjected to a sinusoidal force. The amplitude of forcing is \( F_0 = 40 \text{N} \) and the damping ratio is 0.05.

a) What will be the total spring force?

Solution:

The system can be demonstrated by the Fig. 1.2.

\[ F = F_0 \sin \omega t \]

where, \( \omega \) is the frequency of excitation.

Weight of the system \( W = 356 \text{ N} \)
Mass of the system \( m = 36.29 \text{ kg} \)
Harmonic exciting force \( F_0 = 40 \text{ N} \)
Stiffness of the spring (Spring constant) \( k = \quad \) 
Damping factor i.e, the ratio of actual damping \( c \) to critical damping \( c_c \)
\[ \zeta = \frac{c}{c_c} = 0.05 \]

Actual viscous damping of the system
\[ c = c_c \zeta = 2m \omega_n \zeta \]
Critical viscous damping of the system
\[ c_c = 2m \omega_n \]

The frequency of the excitation
\[ \omega = 20.00 \]

The natural circular frequency of the system
\[ \omega_n = 20.00 \]

At resonance, the frequency ratio,
\[ \frac{\omega}{\omega_n} = 1 \]

The amplitude of oscillation,
\[ Z = \frac{(F_0/k)}{\left(1-(\zeta^2)^{1/2}\right)} \]

i.e, at resonance, \( Z = \frac{(F_0/k)}{2\zeta} \)

Therefore, the spring force at resonance
\[ i.e., kZ = \frac{F_0}{2\zeta} = 400 \text{ N} \]

Problem No. 3: Consider a simple spring-mass-dashpot system with parameters, \( m=4 \text{kg}, k=1.6\times10^3 \text{ N/m} \) and the two cases of damping: (i) \( c=80 \text{N-s/\text{cm}} \) (ii) \( c=320 \text{N-s/\text{m}} \).

a) Study the nature of the free response in each case.

Solution:

Mass of the system \( m = 4 \text{ kg} \)
Stiffness of the spring (Spring constant) \( k = 1600 \text{ N/m} \)
Undamped natural circular frequency of the system \( \omega_n = \sqrt[k/m]{k/m} = 20 \text{ radians/sec.} \)

Case-i) Actual viscous damping of the system
\[ c = c_c \zeta = 2m \omega_n \zeta \]
Damping factor i.e, the ratio of actual damping \( c \) to critical damping \( c_c \)
\[ \zeta = \frac{c}{c_c} = 0.5 \]

The damping is less than critical damping, i.e \( \zeta<1 \). Such a system is called underdamped. Thus, the under damped natural circular frequency in viscously damping vibration equal \( \omega_{ud} = \omega_n \sqrt{1-\zeta^2} \)

Case-ii) Actual viscous damping of the system
\[ c = c_c \zeta = 2m \omega_n \zeta \]
Damping factor i.e, the ratio of actual damping \( c \) to critical damping \( c_c \)
\[ \zeta = \frac{c}{c_c} = 2.0 \]

The damping is more than critical damping, i.e \( \zeta>1 \). Thus, \( z \) decreases as the \( t \) increases but it never changes sign. Such a system is called Overdamped or non-oscillatory. If an initial displacement is given to the system, the mass is pulled back by the springs and dampers absorb all the energy by the time the mass returns to the initial position.
Problem No. 4: A new concrete hall is to be protected from the ground vibrations from an adjacent highway by mounting the hall on rubber blocks. The predominant frequency of the sinusoidal vibration is 40Hz and a motion transmissibility of 0.1 is to be attained at that frequency.

a) Calculate the static deflection required in the rubber blocks. Assuming that these act as linear undamped spring

Solution: \[ F = F_0 \sin \omega t \]

By assuming linear undamped spring,

Transmissibility \[ T_R = \left[ \frac{F_T}{F_0} \right] = \sqrt{1 + \left( \frac{2 \xi \tau}{1-\tau^2} \right)^2} = 0.1 \]

\[ \xi = \frac{\omega}{\omega_n} = 0 \]

The frequency ratio, \[ \tau = \frac{\omega}{\omega_n} = 3.32 \]

The frequency of the excitation \[ f = 40 \text{ Hz}(\text{cps}) \]

The circular frequency of the excitation \[ \omega_n = 2\pi f = 251 \text{ rad/sec} \]

Undamped natural circular frequency of the system \[ \omega_n = \frac{\omega}{\tau} = 75.78 \text{ radians/sec.} \]

Also, \[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{W}{\delta_{stat} / m}} = \sqrt{\frac{mg}{m \delta_{stat}}} = \sqrt{\frac{g}{\delta_{stat}}} \]

Therefore, \[ \delta_{stat} = \frac{g}{\omega_n^2} = 0.17 \text{ cm} \]

Problem No. 5: An electrical motor used as a mechanical drive is mounted at the centre of a light table. The weight of the motor added to the effective weight of the table is 363N. The armature of the rotating part weighs 910N and has an eccentricity =10mm. The table is observed to deflect 3.2mm when the motor is mounted in a free vibration amplitude decreases to 0.2 of the initial value after 6 consecutive cycles. The operating speed of the motor is 900rpm.

a) Calculate the amplitude of vibration of the system at operating frequency.

Solution:

\[ W_1 = 910 \text{ N} \]
\[ m_1 = 92.76 \text{ kg} \]
\[ W_2 = 363 \]
\[ m_2 = 37.00 \text{ kg} \]
\[ M = (m_1 + m_2) = 129.77 \text{ kg} \]
\[ Z_1 = 3.2 \text{ mm} \]
\[ Z_6 = 0.2Z_1 = 0.64 \text{ mm} \]
\[ f = 900 \text{ rpm} \]
\[ \omega = 2\pi f = 94.25 \text{ rad/sec} \]

Damping factor i.e, the ratio of actual damping \( c \) to critical damping \( c_c \)

\[ \xi = \frac{1}{2\pi} \ln(z_1/z_6) = 0.0427 \]

Static deflection \[ \delta_{stat} = 3.20 \text{ mm} \]

Undamped natural circular frequency of the system \[ \omega_n = \frac{g}{\delta_{stat}} = 55.37 \text{ rad/sec.} \]

The frequency ratio, \[ \tau = \frac{\omega}{\omega_n} = 1.70 \]

The eccentricity of the rotating part \[ e = 10 \text{ mm} \]

The amplitude of the vibration \[ Z = \left( \frac{m_1 e}{M} \right) r = \left( \frac{10 \text{ mm}}{129.77 \text{ kg}} \right) \times 2 \times \left( \frac{3.2 \text{ mm}^2}{\left( 1-\tau^2 \right)^2 + \left( 2 \xi \tau \right)^2} \right)^{1/2} = 5.72 \text{ mm} \]
Problem No. 6: A shock absorber is to be designed so that its overshoot is 10 percent of the initial displacement when released.

a) Determine $\xi_1$.
b) If $\xi$ is made equal to 0.5$\xi_1$, what will be the overshoot.

Solution:

Damping factor i.e, the ratio of actual damping $c$ to critical damping $c_c$

$$\xi = \frac{1}{2} \pi n \ln \left( \frac{z_1}{z_2} \right) = 0.3665$$

or, $\xi = \sqrt{\frac{1}{\pi n}} \ln \left( \frac{z_1}{z_2} \right) = 0.3440$

if, $\xi = 0.366 / 2$, i.e, $\xi = 0.183$

i.e, $0.183 = \frac{1}{2} \pi n \ln \left( \frac{z_1}{z_2} \right) \Rightarrow z_1 / z_2 = 3.16$

$\Rightarrow z_2 = 0.316 \ast z_1$

or, $z_2 = 0.334 \ast z_1$

Problem No. 7: A machine mass is 1.95kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic exciting force 24.46N results in resonant amplitude of 1.27 cm with a period of 0.2sec. If the same system is excited by harmonic force of frequency 4cps, what will be the percentage increase in the amplitude of forced vibration when the dash-pot is removed?

Solution:

Mass of the machine

$$m = 1.95 \text{ kg}$$

Period of motion

$$\tau_n = 0.2 \text{ sec}$$

The natural circular frequency of the system

$$\omega_n = \frac{2 \pi}{\tau_n} = 31.42 \text{ rad/sec}$$

Stiffness of the spring (Spring constant)

$$k = m \omega_n^2 = 1925 \text{ N/m}$$

Harmonic exciting force

$$F_0 = 24.46 \text{ N}$$

The amplitude of oscillation,

$$Z = \frac{F_0}{k} \left[ \frac{1}{(1-r^2)^2 + (2 \xi r)^2} \right]^{1/2}$$

i.e, at resonance, $r = 1, Z_{\xi=1} = \frac{F_0}{k} / 2 \xi = 1.27 \text{ cm}$

i.e, at resonance, $\xi = \frac{F_0}{k} / 2Z = 0.50$

When the dashpot is removed,

Harmonic force of frequency

$$f = 4 \text{ Hz(cps)}$$

The circular frequency of the excitation

$$\omega = 2 \pi f = 25.13 \text{ rad/sec}$$

The frequency ratio,

$$\frac{f}{\omega_n} = 0.80$$

The amplitude of oscillation,

$$Z = \frac{F_0}{k} \left[ \frac{1}{(1-r^2)^2 + (2 \xi r)^2} \right]^{1/2}$$

i.e, when $\xi = 0, Z_{\xi=0} = \frac{F_0}{k} / (1-r^2) = 3.53 \text{ cm}$

Therefore, increase in percentage amplitude of forced vibration

$$\left( \frac{Z_{\xi=0} - Z_{\xi=1}}{Z_{\xi=0}} \right) \ast 100 \text{ - } 100 \text{ } \% = 77.98 \text{ % increased}$$
Problem No. 8: A sensitive instrument with mass 113 kg is to be installed at a location where the acceleration is 15.24 cm/sec² at a frequency 20 Hz. It is proposed to mount the instrument on a rubber pad with the following properties: k = 2802 N/cm and \( \xi = 0.10 \). What acceleration is transmitted to the instrument?

Solution:

- Mass of the instrument \( m = 113 \text{ kg} \)
- Harmonic force of frequency \( f = 20 \text{ Hz(cps)} \)
- Period of motion \( \tau = 1/f = 0.05 \text{ sec} \)
- The circular frequency of the excitation \( \omega = 2\pi f = 125.66 \text{ rad/sec} \)
- Stiffness of the spring (Spring constant) \( k = m \omega_n^2 = 2802 \text{ N/cm} \)
- The natural circular frequency of the system \( \omega_n = \sqrt{k/m} = 49.80 \text{ rad/sec} \)
- The frequency ratio, \( r = \omega/\omega_n = 2.52 \)
- Damping factor i.e., the ratio of \( c \) to \( c_c \) \( \xi = c/c_c = 0.1 \)
- Transmissibility \( T_R = \sqrt{1 + (2\xi r)^2 / (1-r^2)} = 0.20 \)
- Acceleration of the location \( a = z_{\text{double dot}} = \omega^2 \sin \omega t = 15.24 \text{ cm/sec}^2 \)
- Acceleration transmitted to the instrument, \( a_T = a \cdot T_R = 3.0 \text{ cm/sec}^2 \)

Problem No. 9: A body weight 60 kg is suspended from a spring, which deflects 1.2 cm under the load. It is subjected to a damping effect adjusted to value 0.2 times that required for critical damping. Find the natural frequency of the undamped, damped vibration and in the later case, determine the ratio of successive amplitudes. If the body is subjected to a harmonic force with a maximum value of 250 N at a frequency equal to twice of its undamped natural frequency, determine the amplitude of forces vibrations and the phase difference with respect to the disturbing forces.

Solution:

- Mass of the body \( m = 60 \text{ kg} \)
- Static deflection \( \delta_{\text{stat}} = 1.2 \text{ cm} \)
- Damping factor i.e., the ratio of \( c \) to \( c_c \) \( \xi = c/c_c = 0.2 \)
- The natural circular frequency of the system \( \omega_n = \sqrt{g/\delta_{\text{stat}}} = 28.59 \text{ rad/sec} \)
- Damped natural frequency \( \omega_{nd} = \omega_n \sqrt{1-\xi^2} = 28.01 \text{ rad/sec} \)
- Ratio of successive amplitudes \( Z_1/Z_2 = \exp (2\pi\xi \sqrt{1-\xi^2}) = 3.61 \)
- The frequency ratio, \( r = \omega/\omega_n = 2 \)
- Harmonic exciting force \( F_0 = 250 \text{ N} \)
- Stiffness of the spring (Spring constant) \( k = m \omega_n^2 = 49050 \text{ N/m} \)
- The amplitude of oscillation, \( Z = (F_0/k) / [(1+r^2)^2 + (2\xi r)^2]^{1/2} = 1.70 \)
- Phase difference between amplitude and disturbing force \( \phi = \tan^{-1} [2\xi r / (1-r^2)] = -14.93^\circ \)
Problem No. 10: A machine foundation supported by a soil layer can be approximated as a mass-spring-dashpot system. The weight of foundation with machine is 800kN. The stiffness constant k is 200x10^3 kN/m and damping constant c is 2340kN-sec/m.

a) Determine critical damping coefficient, damping ratio, logarithmic decrement and damped natural frequency.

b) If the foundation subjected to a vertical force $P_z = P_0 \sin \omega t$ in which $P_0 = 25kN$ and $\omega = 100$ rad/sec, determine the amplitude of vertical vibration of foundation and the maximum dynamic force transmitted to the subgrade.

Solution:

a) Weight of foundation with machine
   
   $W = 800$ kN

   Mass of foundation with machine
   
   $m_1 = 81.55$ kg

   Harmonic exciting force
   
   $P_0 = 250$ kN

   Stiffness of the spring (Spring constant)
   
   $k = m \omega_n^2 = 2.0E+08$ N/m

   Actual viscous damping of the system
   
   $c = c_c \xi = 2m \omega_n \xi = 2.34E+06$ N-sec/m

   Critical viscous damping of the system
   
   $c_c = 2m \omega_n = 2\sqrt{k/m} = 8.08E+06$ N-sec/m

   Damping ratio
   
   $\xi = c/c_c = 0.29$

   Circular frequency of the excitation
   
   $\omega = 2\pi f = 100.00$ rad/sec

   Circular natural frequency of the system
   
   $\omega_n = \sqrt{k/m} = 49.52$ rad/sec

   Damped natural frequency of the system
   
   $\omega_{nd} = \omega_n \sqrt{1-\xi^2} = 47.40$ rad/sec

   $\ln(z_1/z_2) = 2\pi \xi / \sqrt{1-\xi^2} = 1.902$

b) The frequency ratio,

   $r = \omega / \omega_n = 2.02$

   The amplitude of oscillation,

   $Z = (P_0/k) / \sqrt{(1-r^2)}^{1/2} = 0.38$ mm

   Transmissibility

   $T_{FR} = [F_T/F_0] = \text{SORT} (1+(2\xi r)^2) / ((1-r^2)^2+(2\xi r)^2) = 0.47$

   Harmonic exciting force

   $P_0 = 25$ kN

   Force transmitted to the subgrade

   $T_F = [P_0 \cdot T_{FR}] = 11.69$ kN

Problem No. 11: A machine and its foundation weigh 200kN. The spring constant is 120MN/m and damping factor is 20%.

Steady state vibration of the foundation is caused by a disturbing force of $F_d = F_0 \sin \omega t$, in which $F_0=50$kN and the circular frequency of 120 rad/sec. Determine

(a) Undamped natural frequency,

(b) Amplitude of vibration and

(c) Phase angle

Solution:

Harmonic exciting force

$F_0 = 50$ kN

The circular frequency of the excitation

$\omega = 2\pi f = 120$ rad/sec

Weight of machine and its foundation

$W = 200$ kN

Mass of machine and its foundation

$m_1 = 20.39$ kg

Damping factor i.e., the ratio of c to $c_c$

$\xi = c/c_c = 0.20$

Stiffness of the spring (Spring constant)

$k = m \omega_n^2 = 1.2E+08$ N/m

a) The circular natural frequency of the system

$\omega_n = \sqrt{k/m} = 76.72$ rad/sec

b) The frequency ratio,

$\xi = \omega / \omega_n = 1.56$

The amplitude of oscillation,

$Z = (P_0/k) / ((1-r^2)^2+(2\xi r)^2)^{1/2} = 0.26$ mm

c) Phase difference between amplitude and disturbing force

$\phi = \tan^{-1}(2\xi r / (1-r^2)) = -23.40^\circ$
Problem No. 12: A machine is supported on four steel springs for which damping can be neglected. The natural frequency of vertical vibration of the machine-spring system is 3.33 Hz. The machine generates a vertical force $F(t) = F_0 \sin(\omega t)$, the amplitude of the resulting steady state vertical vibration of the system is 5 mm when the machine is operating at 0.33 Hz and 0.6 mm at 10 Hz. Calculate the amplitude of vertical vibration of the machine if the steel springs are replaced by rubber isolators, which provided the same stiffness, but introduce damping $(D)$ equivalent to 0.25 for the system. Comment on the effectiveness of isolators at both frequencies of machine.

Solution:

$$f = 3.33 \text{ Hz(cps)}$$

The frequency of the excitation

$$\omega_n = 2 \pi f = 20.92 \text{ rad/sec}$$

The circular frequency of the excitation

The amplitude of oscillation,

$$Z_1 = 5.00 \text{ mm}$$

The frequency of the excitation

$$f_1 = 0.33 \text{ Hz(cps)}$$

The circular frequency of the excitation

$$\omega_1 = 2 \pi f_1 = 2.07 \text{ rad/sec}$$

The amplitude of oscillation,

$$Z_2 = 0.60 \text{ mm}$$

The frequency of the excitation

$$f_2 = 10.00 \text{ Hz(cps)}$$

The circular frequency of the excitation

$$\omega_2 = 2 \pi f_2 = 62.83 \text{ rad/sec}$$

Damping factor i.e, the ratio of $c$ to $c_c$

$$\xi = c/c_c = 0$$

The frequency ratio,

$$r_1 = \omega_1/\omega_n = 0.099$$

$$r_2 = \omega_2/\omega_n = 3.003$$

The amplitude of oscillation,

$$Z = (F_0/k) / \left[ (1-r^2) + (2 \xi r)^2 \right]^{1/2}$$

$i.e.$, $Z = (F_0/k) / (1-r^2)$

$$=> (F_0/k) = Z_1^{*}(1-r^2) = 4.95 \text{ mm}$$

$$=> (F_0/k) = Z_2^{*}(1-r^2) = -4.81 \text{ mm}$$

Now, damping factor i.e, the ratio of $c$ to $c_c$

$$\xi = c/c_c = 0.25$$

The frequency of the excitation

$$f_1 = 0.33 \text{ Hz(cps)}$$

The circular frequency of the excitation

$$\omega_1 = 2 \pi f_1 = 2.07 \text{ rad/sec}$$

The amplitude of oscillation,

$$Z_1 = (F_0/k) / \left[ (1-r^2) + (2 \xi r)^2 \right]^{1/2} = 4.99 \text{ mm}$$

The amplitude of oscillation,

$$Z_2 = (F_0/k) / \left[ (1-r^2) + (2 \xi r)^2 \right]^{1/2} = 0.61 \text{ mm}$$

**Comment**: Since the amplitude is more or less same as spring system. Therefore, effect of rubber isolator is same as spring systems.

Problem No. 13: A machine of 100 kg mass is supported on springs of total stiffness 700 kN/m, which has an unbalanced rotating element that results in a disturbing force of 350 N at a speed of 3000 rpm. Assume the damping of system is 0.2. Determine

(a) Amplitude of vibration

(b) Transmissibility and

(c) Force transmitted to the foundation-soil system.

Solution:

Harmonic exciting force

$$F_0 = 350 \text{ kN}$$

The frequency of the excitation

$$f = 3000 \text{ Hz(cps)}$$

The circular frequency of the excitation

$$\omega = 2 \pi f = 314 \text{ rad/sec}$$

Mass of machine

$$m_1 = 100 \text{ kg}$$

Damping factor i.e, the ratio of $c$ to $c_c$

$$\xi = c/c_c = 0.20$$

Stiffness of the spring (Spring constant)

$$k = m \omega_n^2 = 7.0 \times 10^5 \text{ N/m}$$

The circular natural frequency of the system

$$\omega_n = \sqrt{k/m} = 83.67 \text{ rad/sec}$$

The frequency ratio,

$$f = \omega/\omega_n = 3.75$$

(a) The amplitude of oscillation,

$$Z = (P_0/k) / \left[ (1-r^2) + (2 \xi r)^2 \right]^{1/2} = 0.038 \text{ mm}$$

(b) Transmissibility

$$T_R = [F_f/F_0] = \text{SORT} \left[ (1+(2 \xi r)^2) / \left[ (1-r^2) + (2 \xi r)^2 \right] \right] = 13.69 \%$$

Force transmitted to the subgrade

$$T_F = [F_0 \times T_R] = 47.90 \text{ kN}$$
Problem No. 14: A two storied building is represented as a lumped mass system (fig.-1) in which $m_1 = 0.5m_2$ and $k_1 = 2k_2$ (the stiffness given represents equivalent lateral stiffness of columns). Determine its natural frequencyes, amplitude of vibration and plot normal mode shapes.

![Fig. 1](image)

The equation of motion can be writted as follows:

$$m_1 \ddot{x}_1 + k_1(x_1 - x_2) = 0$$

$$\Rightarrow m_1 \ddot{x}_1 + k_1x_1 - k_1x_2 = 0 \quad \text{……………… (1)}$$

and

$$m_2 \ddot{x}_2 + k_1(x_2 - x_1) + k_2x_2 = 0$$

$$\Rightarrow m_2 \ddot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 = 0 \quad \text{……………… (2)}$$

From equation (1) & (2) we get,

$$\begin{vmatrix}
    k_1 - m_1 \omega_n^2 & -k_1 \\
    -k_1 & k_1 + k_2 - m_2 \omega_n^2
  \end{vmatrix} = 0$$

Mode Shapes:

![Mode Shapes](image)